Logarithmic scaling and critical collapse in Davey-Stewartson equation

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Support:
NSF 0807131, NSF 1004118, NSF 141214, NSF 1814619

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  formation of singularity in a finite time

- Collapse: blow-up with the contraction of the
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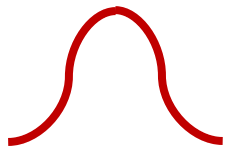

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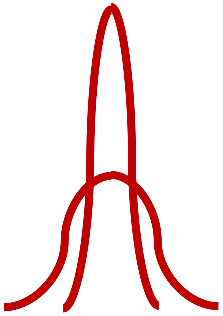

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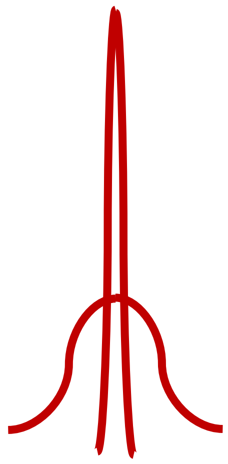

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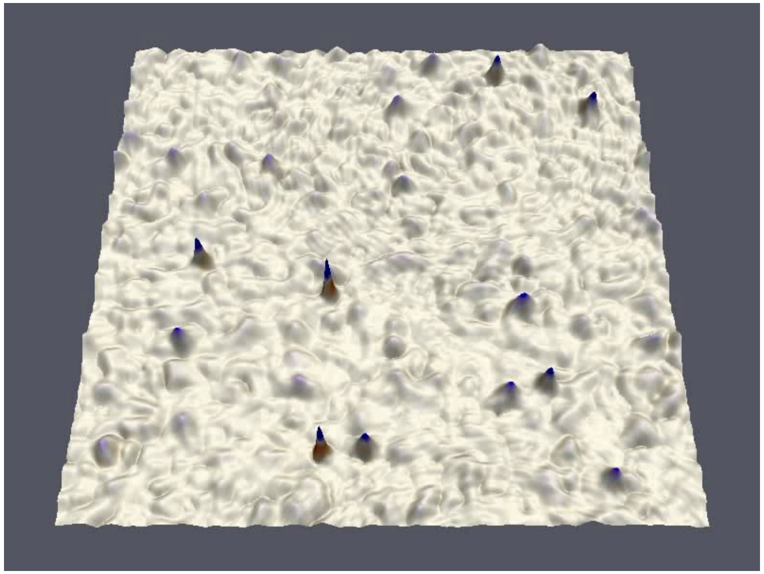
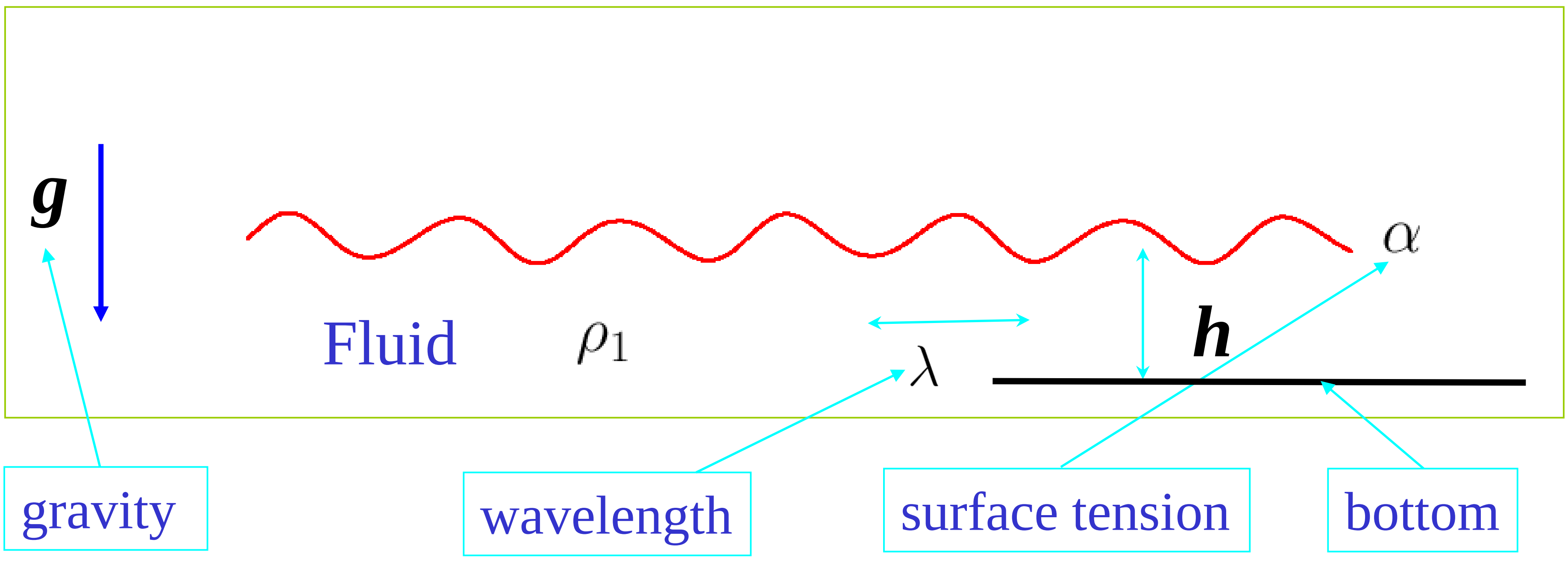


Multiple collapses (filamentation) of laser beam

Dynamics of water waves of finite depth $\boldsymbol{h}$

Potential flow: $\mathbf{v} = \nabla\Phi$



$\psi e^{ikx - i\omega_k t} + c.c.$ - wave amplitude (envelope)

$\omega_k^2 = (gk + \alpha k^3) \tanh(kh)$ - dispersion relation

Infinite depth $\boldsymbol{h} \to \infty$

$$k^2\alpha/g > 1/2 \quad \Rightarrow \quad \lambda \lesssim 2.4\text{cm}$$

$\Rightarrow$ Focusing 2D Nonlinear Schrödinger Equation (NLSE)

$$i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi = 0$$

$$k^2\alpha/g < 1/2 \quad \Rightarrow \quad \lambda \gtrsim 2.4\text{cm}$$

$\Rightarrow$ Hyperbolic 2D NLSE[1]

$$i\psi_t + (\partial_x^2 - \partial_y^2)\psi + |\psi|^2\psi = 0$$

[1]V.E. Zakharov (1968).

Finite depth *h*

Davey-Stewartson equation[1,2] (DSE), also called by Benney-Roskes equation[3,4]

$$iA_\tau + \lambda A_{\xi\xi} + \mu A_{\eta\eta} = \chi|A|^2 A + \chi_1 A\Phi_\xi,$$

$$\alpha\Phi_{\xi\xi} + \Phi_{\eta\eta} = -\beta(|A|^2)_\xi,$$

Φ results from the soft mode of the motion of the entire depth of fluid

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[1]A. Davey and K. Stewartson (1974).

[2]D.J. Benney and G.J. Roskes (1969).

[3]V.D. Djordjevic and L.G. Redekopp (1977).

[4]M.J. Ablowitz and H. Segur (1979).

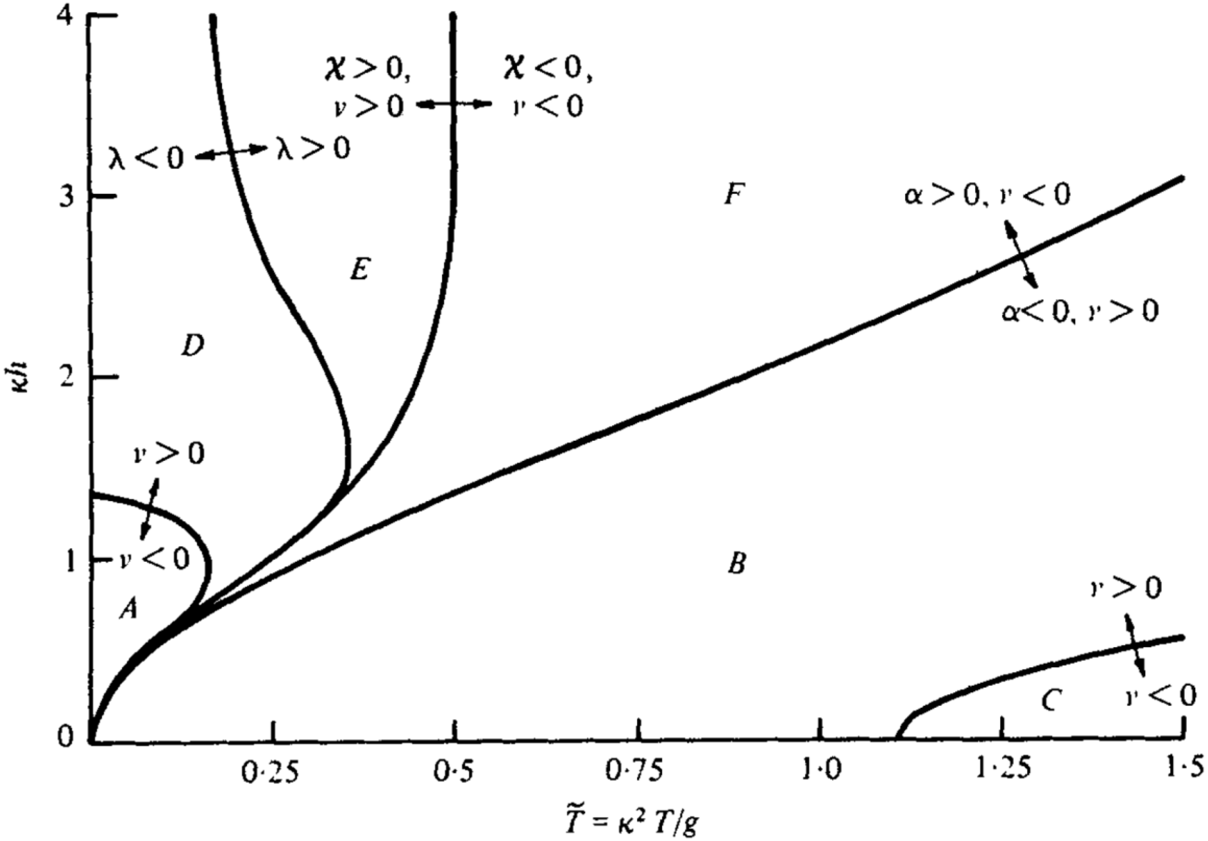

FIGURE 1. Map of parameter space, showing where the coefficients in (2.25) change sign. The dynamics of wave evolution is different in each region.

$$iA_\tau + \lambda_\infty A_{\xi\xi} + \mu_\infty A_{\eta\eta} = \chi_\infty |A|^2 A, \qquad (2.25)$$

$$\lambda_\infty = -\frac{\omega_0}{8\omega}\left(\frac{1-6\tilde{T}-3\tilde{T}^2}{1+\tilde{T}}\right),$$

$$\mu_\infty = \frac{\omega_0}{4\omega}(1+3\tilde{T}),$$

$$\chi_\infty = \frac{\omega_0}{4\omega}\frac{8+\tilde{T}+2\tilde{T}^2}{(1-2\tilde{T})(1+\tilde{T})}.$$

M.J. Ablowitz and H. Segur, JFM, **92**, 691-715 (1979).

Davey-Stewartson Eq. in maximally rescaled coordinates:

$$i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - \mu\psi\partial_x\phi = 0,$$
$$\partial_x^2\phi + \nu\partial_y^2\phi = \partial_x(|\psi|^2), \qquad \nu > 0$$

Integrable cases (hyperbolic-elliptic and elliptic-hyperbolic):

DS I:

$$i\psi_t + (-\partial_x^2 + \partial_y^2)\psi - |\psi|^2\psi + 2\psi\partial_x\phi = 0,$$
$$\partial_x^2\phi + \partial_y^2\phi = \partial_x(|\psi|^2),$$

DS II:

$$i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - 2\psi\partial_x\phi = 0,$$
$$\partial_x^2\phi - \partial_y^2\phi = \partial_x(|\psi|^2).$$

Focus on elliptic-elliptic case:

$$i\psi_t + (\partial_x^2 + \partial_y^2)\psi + |\psi|^2\psi - \mu\psi\partial_x\phi = 0,$$
$$\partial_x^2\phi + \nu\partial_y^2\phi = \partial_x(|\psi|^2), \qquad \nu > 0$$

The Hamiltonian

$$H = \int |\nabla\psi|^2 d\mathbf{r} - \frac{1}{2}\int |\psi|^4 d\mathbf{r} + \frac{\mu}{2}\int (\phi_x^2 + \nu\phi_y^2) d\mathbf{r}$$

Virial theorem

$$\frac{d^2}{dt^2}\int (x^2+y^2)|\psi|^2 d\mathbf{r} = 8H \qquad \Rightarrow$$ Collapse for $H<0$[1-6]

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[1] M.J. Ablowitz and H. Segur, JFM, **92**, 691-715 (1979).
[2] G.C. Papanicolaou, C. Sulem, P.L. Sulem, X.P. Wang, Physica D, **72**, 61 (1994)
[3] M.J. Ablowitz, G. Biondini, S. Blair, Phys. Lett. A **236** (1997) 520.
[4] M.J. Ablowitz, G. Biondini, S. Blair, Phys. Rev. E **63** (2001) 605.
[5] M.J. Ablowitz, I. Bakirtas and B. Ilan (2005).
[6] M.J. Ablowitz, I. Bakirtas and B. Ilan (2005).

Critical NLSE collapse

$$|\psi| \sim \frac{1}{L(t)} R\left(\frac{r}{L(t)}\right), \quad L(t) \propto (t_c - t)^{1/2}$$

ground state soliton of NLSE

Critical collapse in Davey-Stewartson Eq (DSE):

$$|\psi| \sim \frac{1}{L(t)} R\left(\frac{x}{L(t)}, \frac{y}{L(t)}\right), \quad L(t) \propto (t_c - t)^{1/2}$$

ground state soliton of DSE

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[1] M.J. Ablowitz, I. Bakirtas and B. Ilan (2005).

Collapses in critical NLSE and critical Keller-Segel equation (KSE)

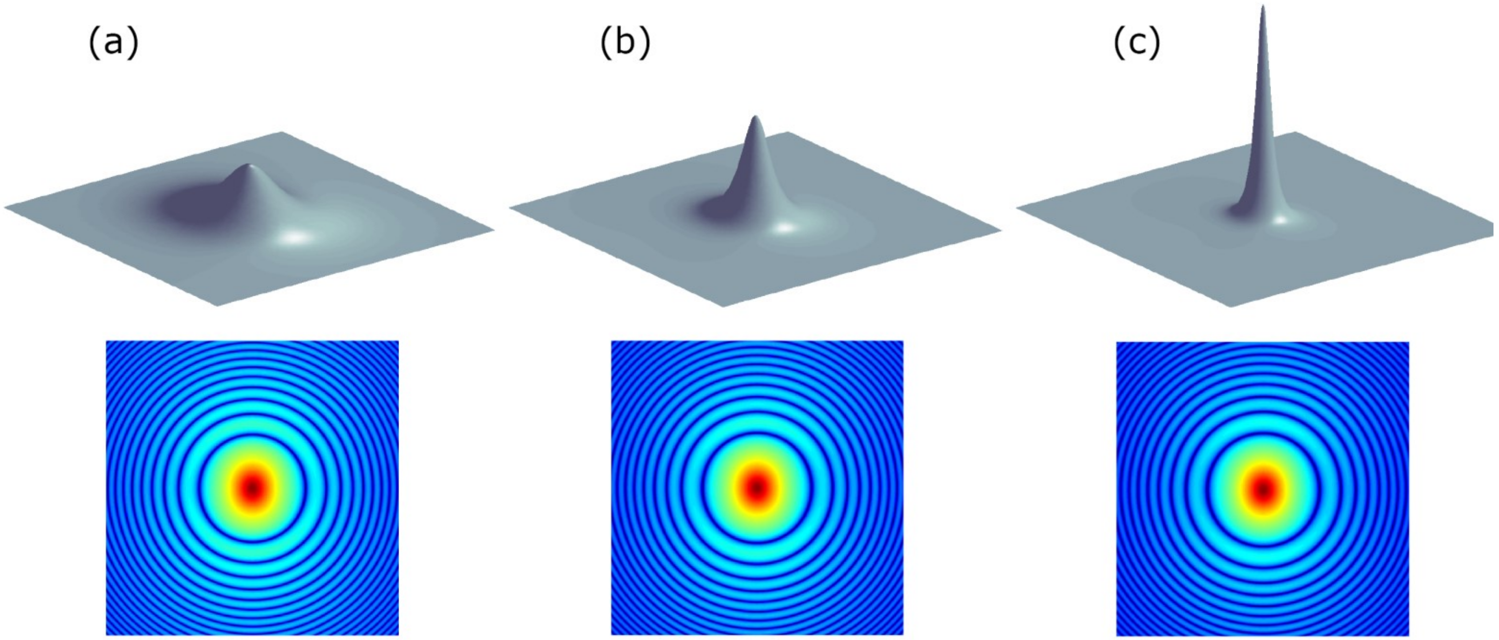
2D NLSE

$$\boxed{i\frac{\partial}{\partial t}\psi + \triangle\psi + |\psi|^2\psi = 0}$$

2D KSE

$$\boxed{\begin{aligned}\frac{\partial\rho}{\partial t} &= \Delta\rho - \nabla\left(\rho\nabla c\right)\\ 0 &= \Delta c + \rho.\end{aligned}}$$

Simulations of Davey-Stewartson Eq (DSE) anisotropic collapse:



Critical collapse in Davey-Stewartson Eq (DSE):

Blow up variables $$p \equiv \frac{x}{L(t)}, \quad q \equiv \frac{y}{L(t)}, \quad \tau = \int_0^t \frac{dt'}{L(t')^2}$$

and lens transform

$$\boxed{\psi(\mathbf{r}, t) = \frac{1}{L(t)} V(p, q, \tau) e^{i\tau + iL(t)_t L(t)(p^2+q^2)/4}}$$

$\Rightarrow$ Davey-Stewartson Eq. transforms into

$$\boxed{i\partial_\tau V + \nabla^2_{p,q} V - V + |V|^2 V + \frac{\beta}{4}(p^2 + q^2)V - \mu V \frac{\partial_p^2}{\partial_p^2 + \nu \partial_q^2}|V|^2 = 0,}$$

where $\boxed{\beta = -L^3 L_{tt}}$ - adiabatically slow small parameter $\beta \ll 1$

and $\nabla^2_{p,q} \equiv \partial_p^2 + \partial_q^2$

Why $\boxed{\beta = -L^3 L_{tt}}$ adiabatically slow small parameter?

$a = -L(t)\partial_t L(t) > 0$

$$L(t) = \sqrt{t_c - t}\, f(ln(t_c - t)) \implies a = -LL_t = \frac{\sqrt{t_c - t}}{2\sqrt{t_c - t}} f^2 + \frac{t_c - t}{tc - t} f f' = \frac{f^2}{2} + f f'$$

Logarithmically slow functions

Then $\beta = -L^3 L_{tt} \simeq -a^2$ is also adiabatically slow function of time

Looking for solution in the form

$$V = V_0 + V_1 + \ldots$$

In adiabatic approximation of slow β

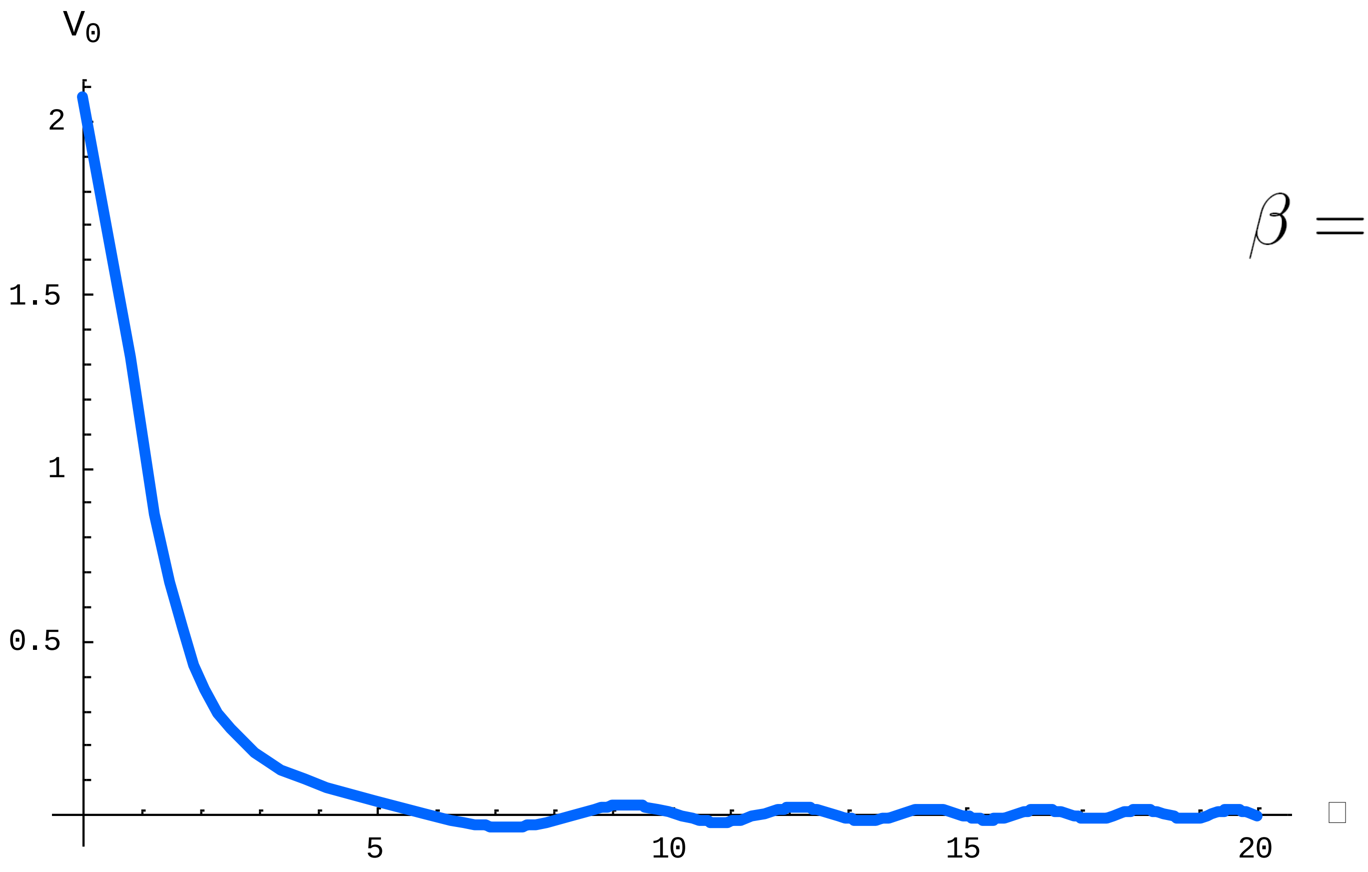
$$\nabla^2_{p,q} V_0 - \quad V_0 + |V_0| V_0 + \frac{\beta}{4}(p^2 + q^2) V_0 - \mu V_0 \frac{\partial_p^2}{\partial_p^2 + \nu \partial_q^2} |V_0|^2 = 0$$

Tail minimization principle: during collapse dynamics system dynamically select collapsing solution with minimal tail amplitude

Then we look for $\boldsymbol{V_0}$ with the minimal tail

NLSE: In adiabatic approximation of slow β minimizing tails by shooting method:

$$\nabla^2 V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4}\rho^2 V_0 = 0$$

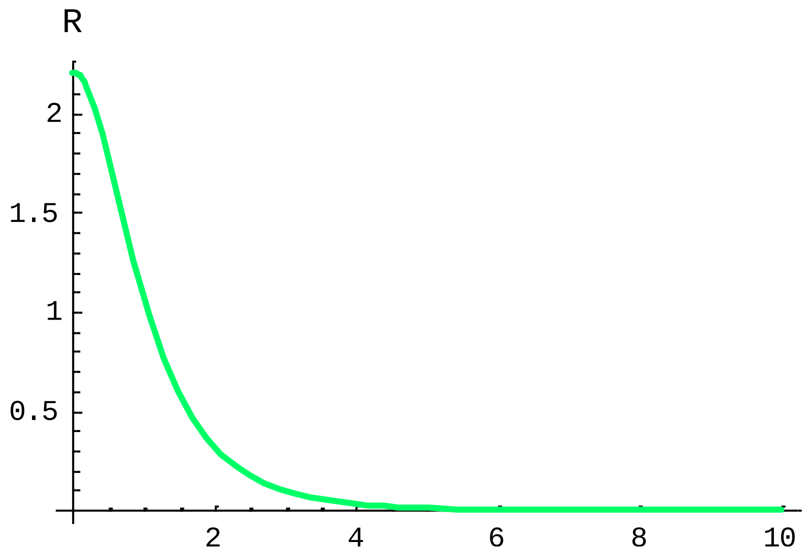


$$\beta = 0.2$$

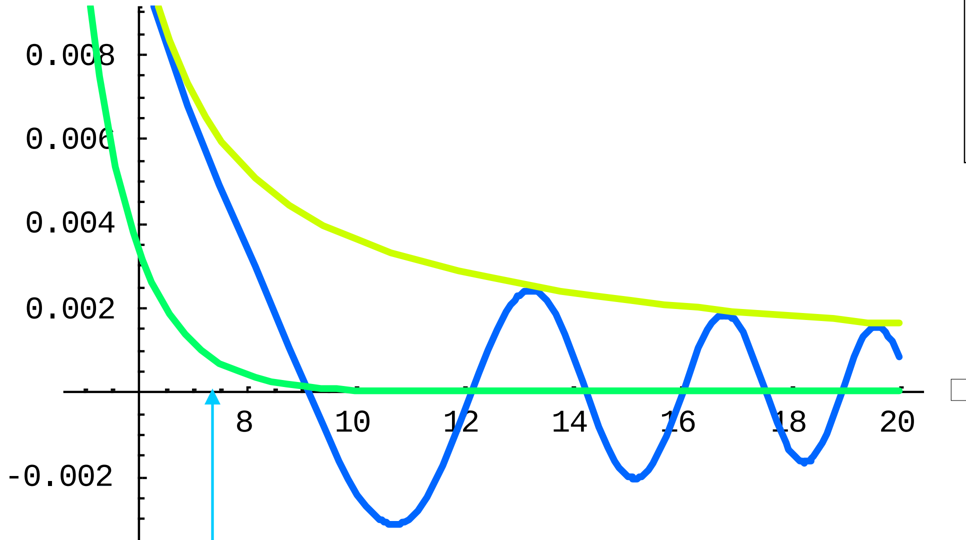
Approximation through ground state soliton $R(\rho)$

$$V_0 = R(\rho) + \beta \frac{\partial V_0}{\partial \beta}\Big|_{\beta=0} + O(\beta^2)$$

$$-R + \nabla^2 R + R^3 = 0$$



NLSE: Full solution *V* match the envelope of V_0 of in the tail:



$|V|$ - from numerics

V_0 – soliton with β

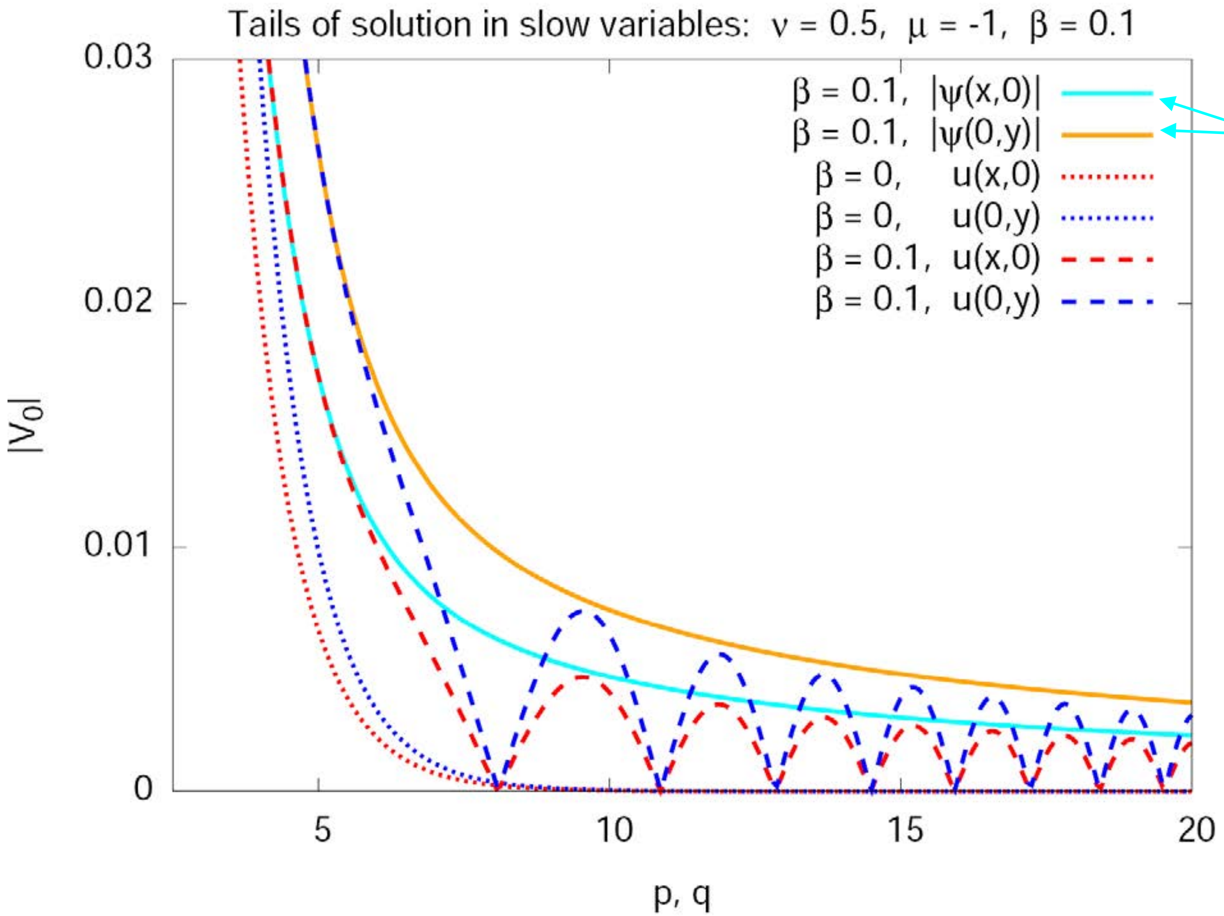
R – ground state soliton with $\beta = 0$

$\rho_b \simeq 7.4$

$\beta = 0.073$

$V_0 \simeq |V|$ to the left from ρ_b

DSE: Full solution *V* match the envelope of V_0 of in the tail along both spatial directions



Time dependent numerics shown in two directions

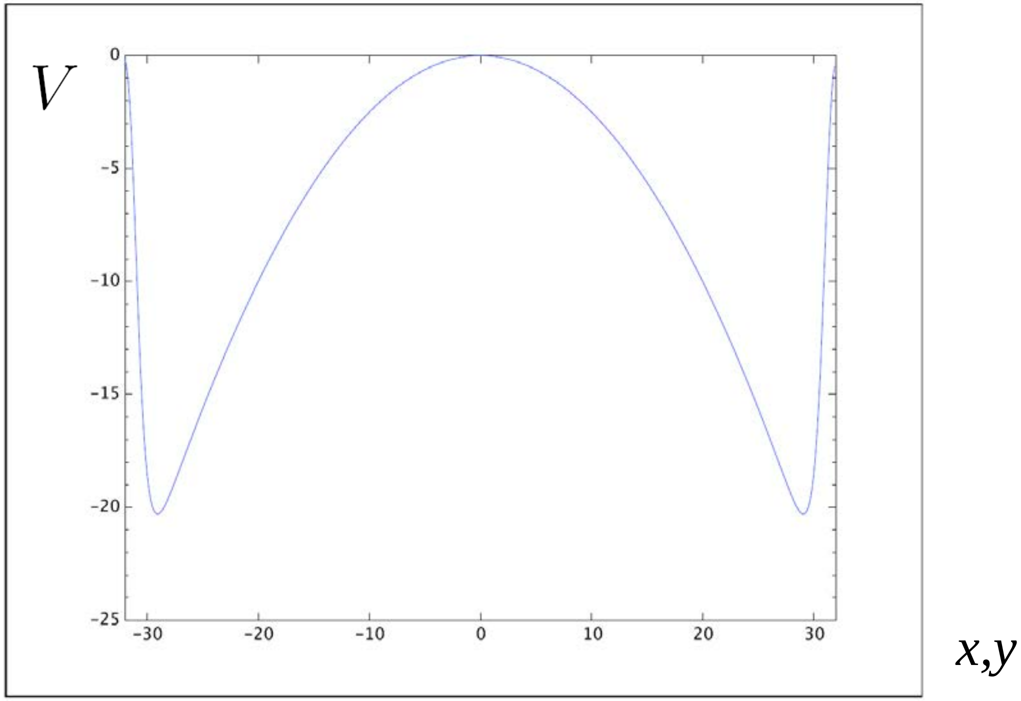
How V_0 was obtained? (cannot use shooting in 2D elliptic problem)

Use Newton-conjugate-gradient method (J. Yang , 2009) combined with the correct choice of the asymptotic at infinity:

$$\nabla^2 u - \lambda u + u^3 - \mu\phi u - V(x,y)u = 0$$

$$\phi_{xx} + \nu\phi_{yy} = (u^2)_{xx}$$

Here $V = -\frac{1}{4}\beta(x^2+y^2)$ for the infinite domain but instead cutoff at large distances for the finite domain as follows:



Numerics: solve by iterations $\mathbf{u}_{n+1}(\mathbf{x}) = \mathbf{u}_n(\mathbf{x}) + \Delta\mathbf{u}_n(\mathbf{x})$

to satisfy the nonlinear system $\mathbf{L}_0\mathbf{u}(\mathbf{x}) = 0$

$$L_0^{(u)}u \;=\; \nabla^2 u - (V - u^2 + \lambda)u - \mu\phi u,$$

$$L_0^{(\phi)}\phi \;=\; \phi_{xx} + \nu\phi_{yy} - (u^2)_{xx},$$

using a linearization about a current iteration $\mathbf{L}_0\mathbf{u}_n + \mathbf{L}_{1n}\Delta\mathbf{u}_n = 0$
with the linearization operator

$$L_1^{(u)}\delta u \;=\; \nabla^2\delta u - (V - 3u^2 + \lambda)\delta u - \mu\phi\delta u - \mu u\delta\phi,$$

$$L_1^{(\phi)}\delta\phi \;=\; (\delta\phi)_{xx} + \nu(\delta\phi)_{yy} - (2u\delta u)_{xx}.$$

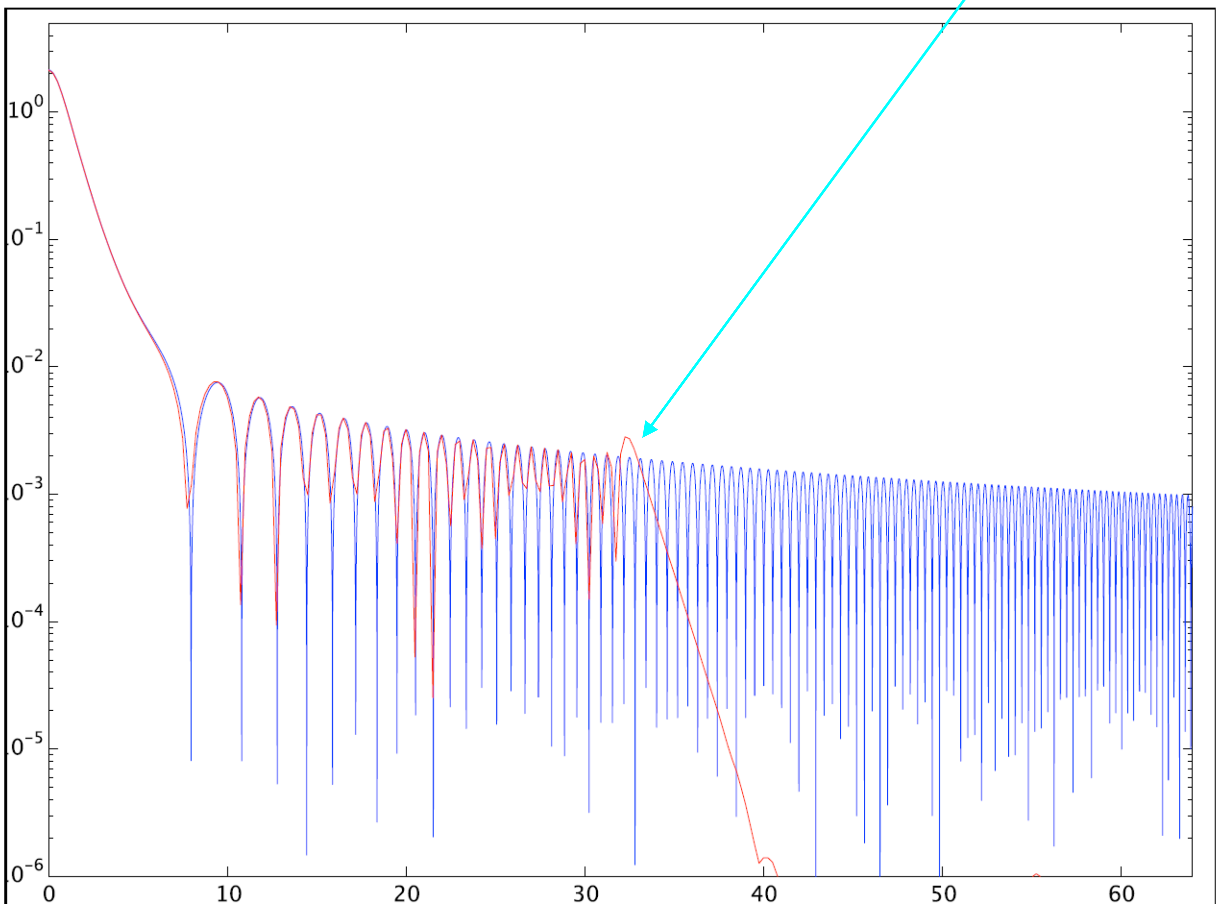
At each iteration step the linear system $\mathbf{L}_0\mathbf{u}_n + \mathbf{L}_{1n}\Delta\mathbf{u}_n = 0$

is solved for $\Delta\mathbf{u}_n$ **by Conjugate gradient method (allows fast FFT-type solved with *N Log N* operations vs. regular solvers**

DSE: In adiabatic approximation of slow β minimizing tails by Newton-conjugate-gradient method (J. Yang , 2009) combined with the correct choice of the asymptotic at infinity by optimizing the cutoff distance of the potential *V* to decrease an artificial bump beyond oscillations:

$$\nabla^2_{p,q}V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4}(p^2+q^2)V_0 - \mu V_0 \frac{\partial_p^2}{\partial_p^2+\nu\partial_q^2}|V_0|^2 = 0$$

V_0



p or ***q***

NLSE: How to extract *L(t)* and β from simulations:

The analysis of Taylor series solution of

$$\nabla^2_{p,q} V_0 - \quad V_0 + V_0^3 + \frac{\beta}{4}(p^2+q^2)V_0 - \mu V_0 \tilde{\phi} = 0,$$

$$(\partial_p^2 + \nu \partial_q^2)\tilde{\phi} = \partial_p^2 V_0^2, \qquad \tilde{\phi} \equiv L\partial_p \phi = L^2 \partial_x \phi$$

vs. time-dependent numerics at $|\boldsymbol{p}|, |\boldsymbol{q}| \ll \mathbf{1}$.

Use that $$|\psi(\mathbf{0}, t)| = \frac{1}{L(t)} V_0(0, 0, \beta).$$

$$\Rightarrow \qquad L = \qquad |\psi|^{1/2} \left(\frac{1}{|\psi|^3 + |\psi|_{xx} + |\psi|_{yy} - \mu |\psi| \partial_x \phi} \right)^{1/2} \Big|_{\mathbf{r}=0}.$$

Then β is found from the implicit equation

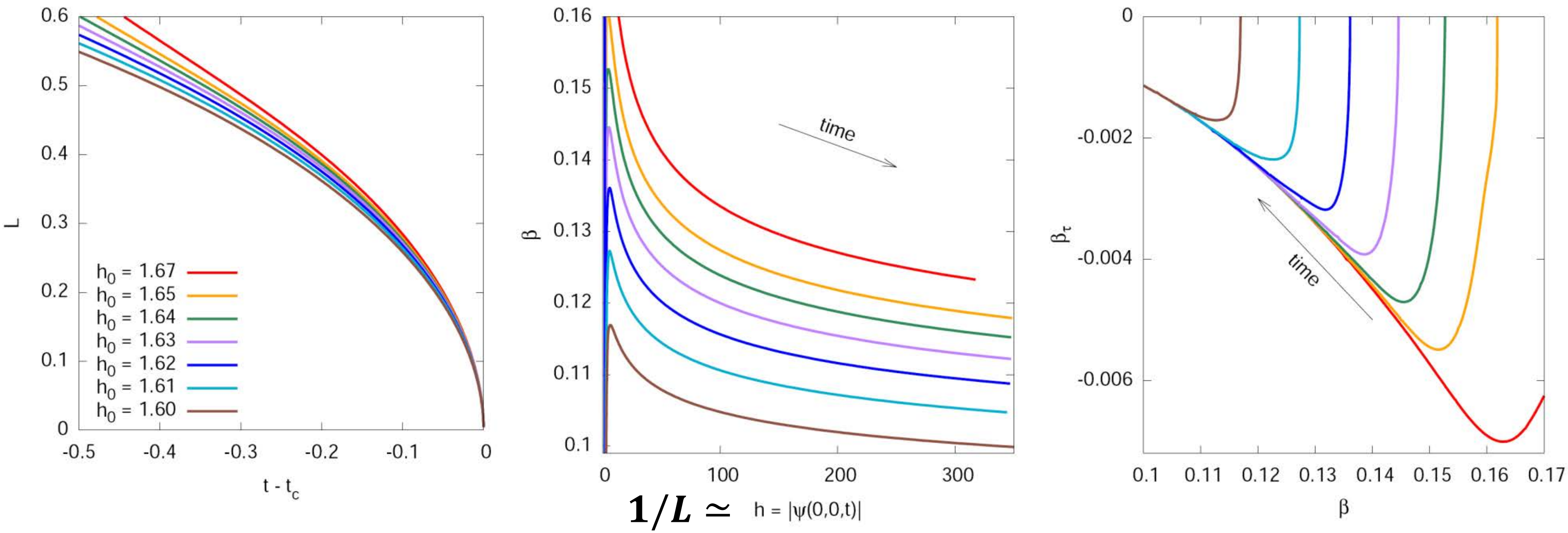
$$\psi(r=0,t) = \tfrac{1}{L(t)} V_0(\beta, \rho = 0)$$

for each given $L(t)$ and $\psi(r=0,t)$

Recovery of $L(\tau)$ and $\beta(\tau)$ from numerics:

$L(\tau)$ and $\beta(\tau)$ are not universal but

$\beta_\tau(\beta)$ is universal (different colors are different initial conditions):



Look at

$$\nabla^2_{p,q}\tilde{V}_0 - V_0 + |\tilde{V}_0|^2\tilde{V}_0 + \frac{\beta}{4}(p^2+q^2)\tilde{V}_0 - \mu\tilde{V}_0\frac{\partial_p^2}{\partial_p^2+\nu\partial_q^2}|\tilde{V}_0|^2 - i\nu(\beta)\tilde{V}_0 = 0$$

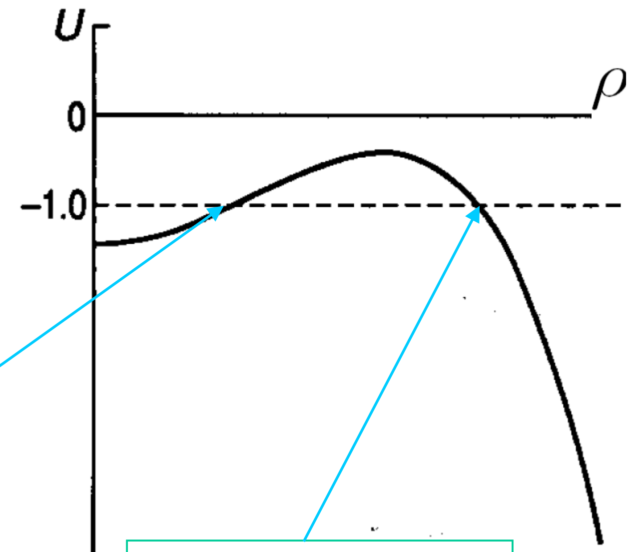
as the Schrodinger equation with the effective potential *U*:

$$U = -|\tilde{V}_0|^2 - \mu\tilde{V}_0\frac{\partial_p^2}{\partial_p^2+\nu\partial_q^2} - \frac{\beta}{4}(p^2+q^2)\tilde{V}_0, \qquad \rho = (p^2+q^2)^{1/2}$$

and complex eigenvalue *E*:

$$E = -1 - i\nu(\beta)$$

$\Rightarrow$ **2 turning points ρ_a and ρ_b of WKB:**



$$\rho_a \sim 1$$

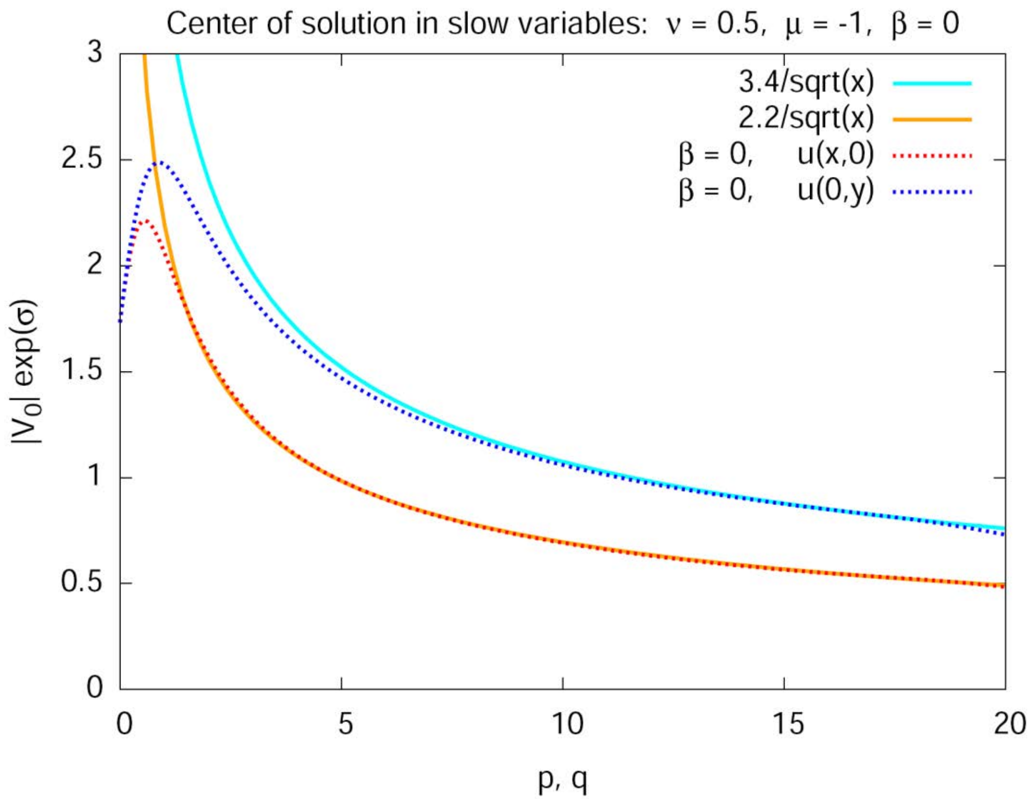
$$\rho_b \simeq \frac{2}{\beta^{1/2}}$$

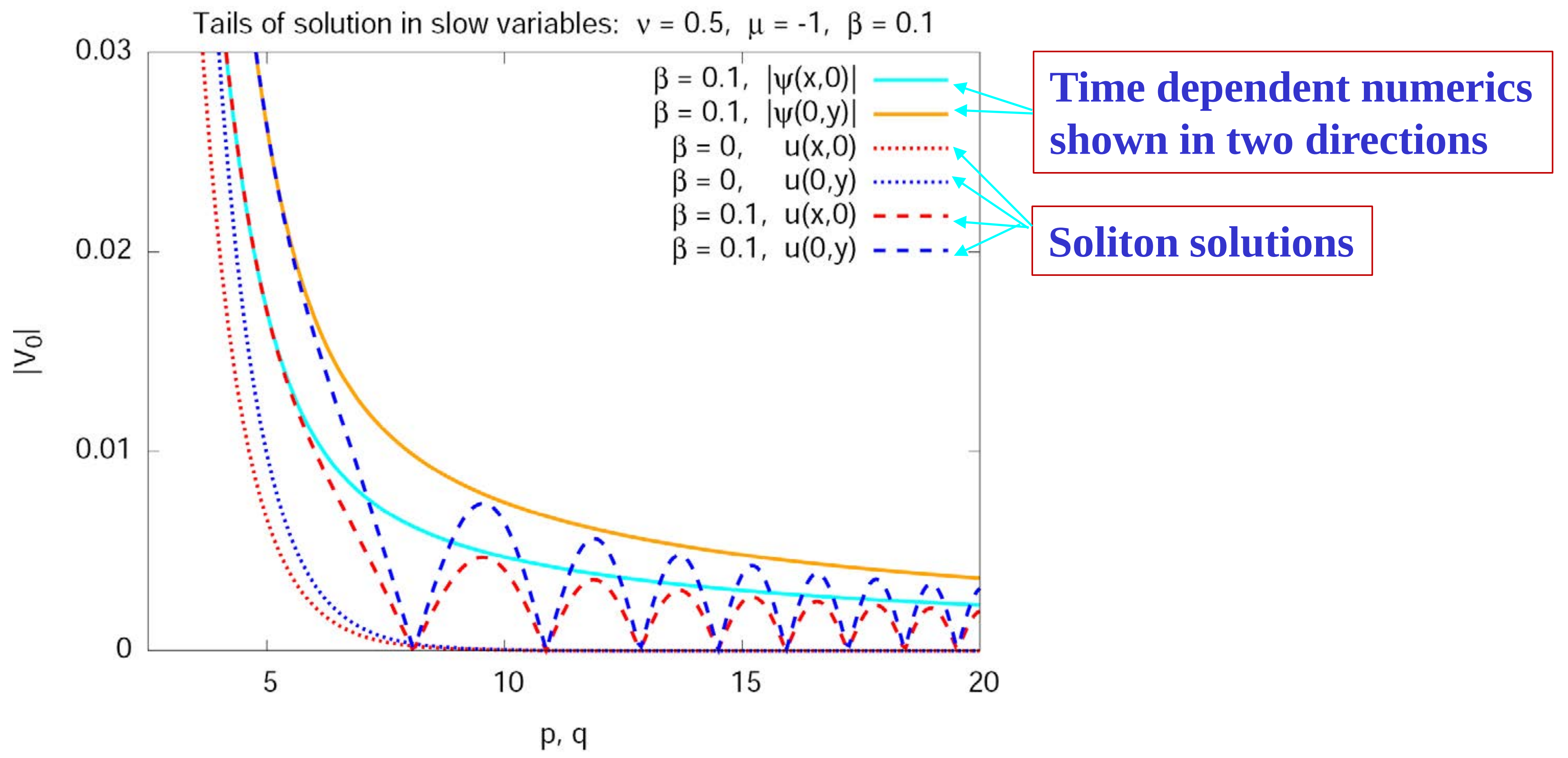
Matching of WKB solution to the right from the left of the left turning point to the asymptotic $R_0 = \frac{A_R(\theta)}{\sigma^{1/2}} \exp\left[-\sigma\right], \;\; \sigma \equiv (p^2+q^2)^{1/2}$

Here A_R depends on spatial angle Θ as well as on ν and μ as $A_R(\Theta) = A_0 + A_2 \cos(2\Theta)$ as follows from multipole expansion for

$$\mu V \frac{\partial_p^2}{\partial_p^2 + \nu \partial_q^2} |V|^2$$

Example:





Tails of solution in slow variables: ν = 0.5, μ = -1, β = 0.1
0.03
0.02
0.01
0
|V0|
5
10
15
20
p, q
β = 0.1, |ψ(x,0)|
β = 0.1, |ψ(0,y)|
β = 0, u(x,0)
β = 0, u(0,y)
β = 0.1, u(x,0)
β = 0.1, u(0,y)
Time dependent numerics shown in two directions
Soliton solutions

Oscillating tail is given by the linear combination of confluent hypergometric functions of the first and second kinds:

$$c_1 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}\ {}_1F_1(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2)+c_2 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}U(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2).$$

Matching asymptotics and using WKB give that

$$V_0(\beta,\rho)=\frac{2^{1/2}A_R}{\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\cos\left[\frac{\beta^{1/2}}{4}\rho^2-\beta^{-1/2}\ln\rho+\phi_0\right],\qquad \rho\gg\rho_b$$

Here A_R **is determined by the asymptotic of ground state soliton** $R_0(\rho)=\frac{A_R}{\rho^{1/2}}e^{-\rho},\quad \rho\gg 1$

$\Rightarrow$ **Asymptotics of complex solution**

$$V(\beta,\rho)=\frac{2^{1/2}A_R}{-\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\exp\left[i\frac{\beta^{1/2}}{4}\rho^2-i\beta^{-1/2}\ln\rho-i\phi_0\right],\quad \rho\gg\rho_b.$$

Introducing the number of particles to the left of the second turning point

$$N_b = \int\limits_{r<\rho_b L} |\psi|^2 d\mathbf{r} = 2\pi \int\limits_{\rho<\rho_b} |V|^2 \rho d\rho.$$

and balancing the flux of particles to oscillating tails with the loss in N_b:

$$\frac{dN_b}{d\tau} = \beta_\tau \frac{dN_b}{d\beta} \quad \text{=-flux}$$

$\Rightarrow$ ODE system qualitatively similar to NLSE

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

Here $$\frac{dN_b}{d\beta} = M\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]$$

Compare with old basic ODE system of the standard theory

$$\begin{cases} \beta_\tau = -\tilde{M} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

Asymptotic solution near collapse time t_c :

$$\boxed{L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}}$$

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[1]G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992); V. F. Malkin (1993).

Conclusion and future directions

- Critical collapse of DSE results in log-log scaling as in NLSE

- Tail minimization principle ensures matching of time dependent simulations with self-similar-like soliton solution with finite β

- Going beyond leading order scaling similar to NLSE is possible and will be done as the next step similar to Ref.[1]

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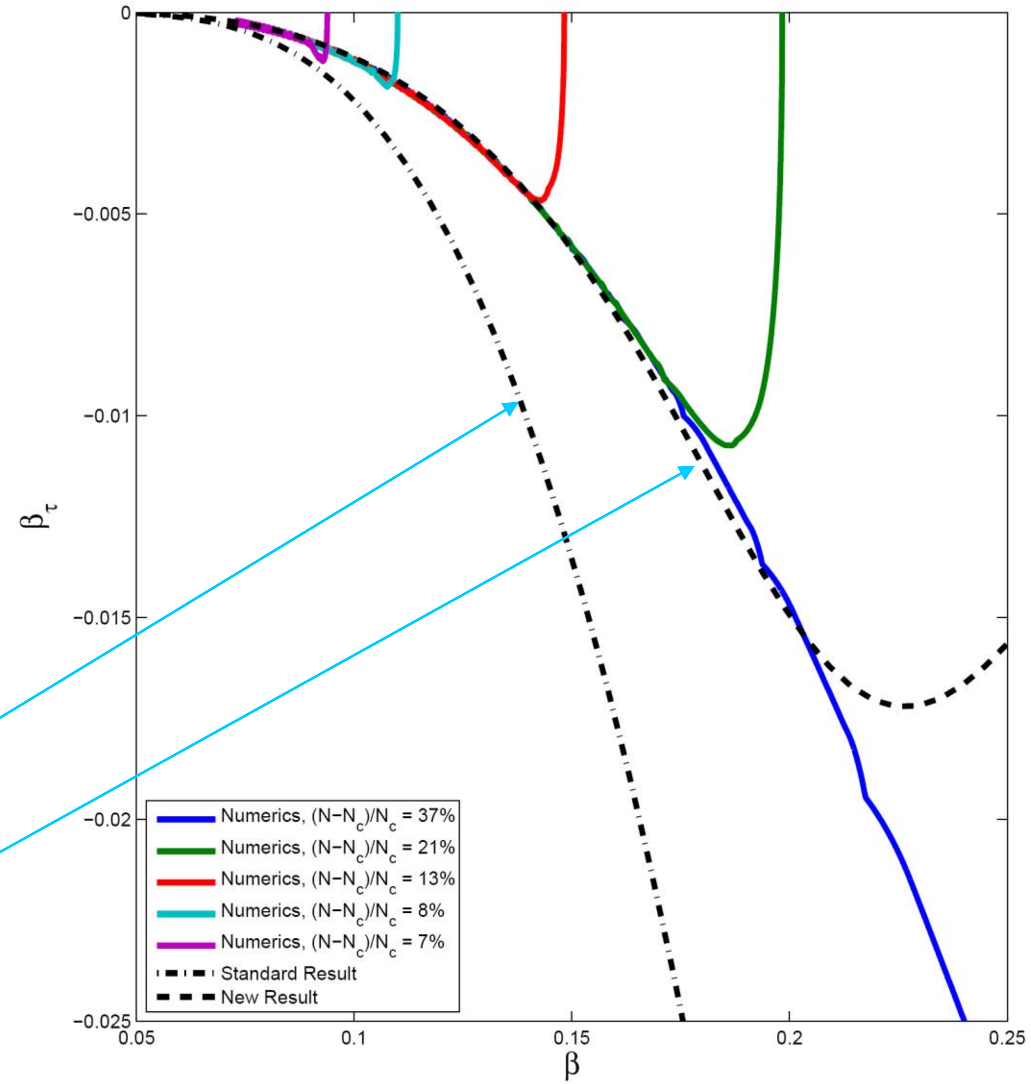
[1]P.M. Lushnikov, S.A. Dyachenko and N. Vladimirova. Physical Review A, v. 88, 013845 (2013).

Returning to previous Figure

L(t) **is not universal but**
$\boldsymbol{\beta_\tau(\beta)}$ **is universal:**



$$\beta_\tau = -\tilde{M} \exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

$$\beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

Finding asymptotic of a new basic ODE system

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

$$-\ln\frac{L}{L_0} = \frac{2\pi^3 e^x}{\tilde{M}}\Bigg[\frac{1}{x^4} + \frac{4}{x^5} + \frac{20+\pi^2 c_1}{x^6} + \frac{120+6\pi^2 c_1}{x^7} + \frac{840+42\pi^2 c_1+\pi^4 c_2}{x^8} + \frac{6720+336\pi^2 c_1+8\pi^4 c_2}{x^9}$$
$$+\frac{60480+3024\pi^2 c_1+72\pi^4 c_2+\pi^6 c_3}{x^{10}} + \frac{604800+30240\pi^2 c_1+720\pi^4 c_2+10\pi^6 c_3}{x^{11}}$$
$$+\frac{6652800+332640\pi^2 c_1+7920\pi^4 c_2+110\pi^6 c_3+\pi^8 c_4}{x^{12}} + \frac{79833600+3991680\pi^2 c_1+95040\pi^4 c_2+1320\pi^6 c_3+12\pi^8 c_4}{x^{13}}$$
$$+\frac{1037836800+51891840\pi^2 c_1+1235520\pi^4 c_2+17160\pi^6 c_3+156\pi^8 c_4+\pi^{10} c_5}{x^{14}} + O\left(\frac{1}{x^{15}}\right)\Bigg]$$

$$x = \frac{\pi}{\beta^{1/2}}$$

$$\tau = \int_0^t \frac{dt'}{L^2(t')} \qquad \Rightarrow$$

$$t_c - t = \int_t^{t_c} dt = \int_\tau^\infty L^2 d\tau = \int_\beta^0 L^2 \frac{d\tau}{d\beta} d\beta$$

$$= -\int_\beta^0 L^2 \frac{1}{\tilde{M}} \left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right] \exp\left[\frac{\pi}{\beta^{1/2}}\right] d\beta$$

Using $\beta(L)$ from the inversion of previous expression and inverting that equation

$$\Rightarrow$$

Asymptotic of new basic ODE system

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

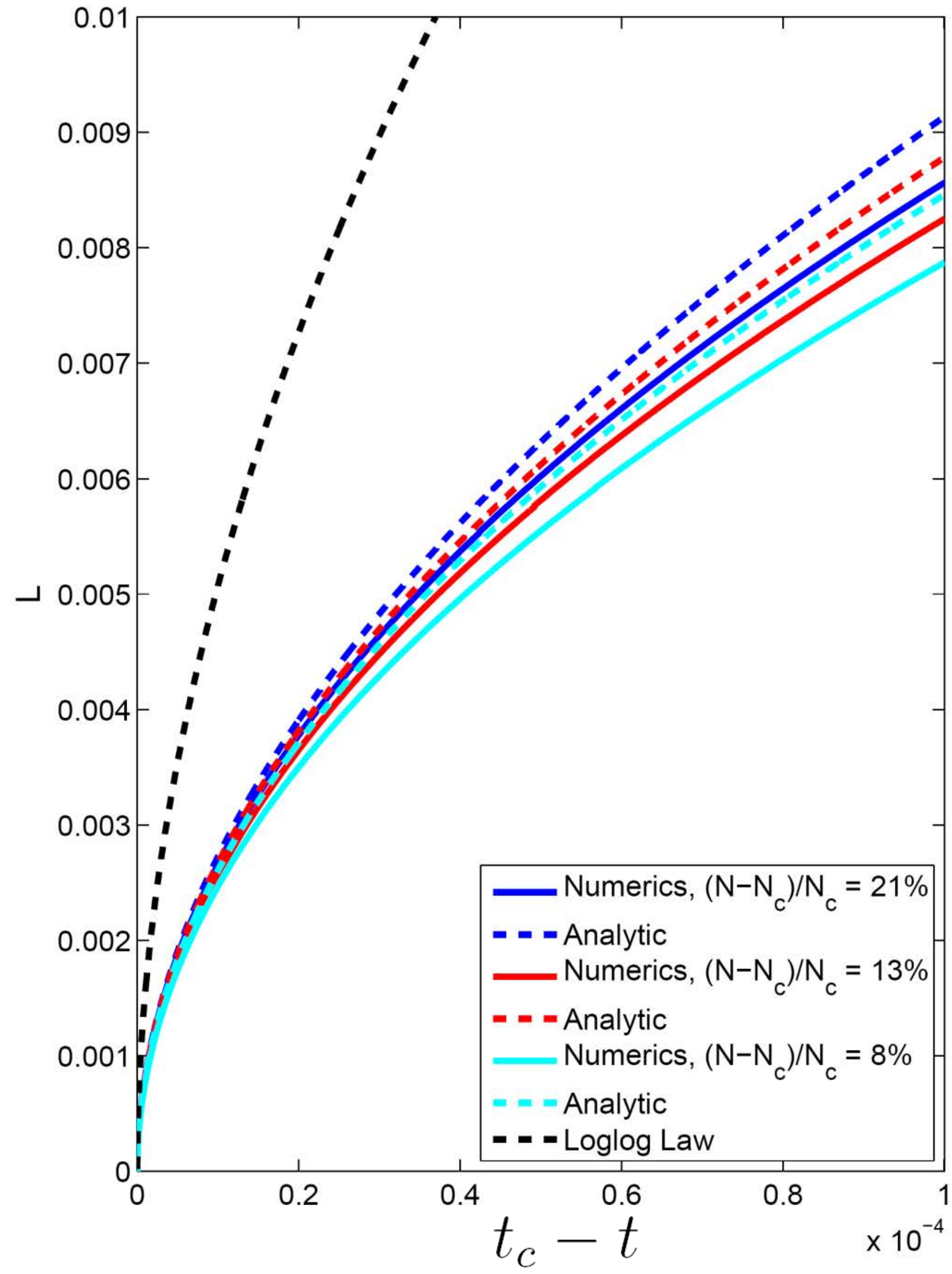
$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A}\right)^{1/2}\left[1 + \frac{2(1 + 4\ln 3 - 4\ln\ln A)}{(\ln A)^2}\right.$$
$$\left. + \frac{14 - 48\ln\ln A + 48(\ln\ln A)^2 + 48\ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} + O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\right]$$

$$A = -3^4\frac{\tilde{M}}{2\pi^3}\ln\left[[2\pi(t_c - t)]^{1/2}\frac{e^{-a_0}}{L(z_0)}\right], \quad \tilde{M} = 44.773\ldots, \quad \beta_0 = \beta(t_0), \ c_1 = 4.793\ldots, c_2 = 52.37\ldots$$

$$a_0 = \frac{e^{\frac{\pi}{\sqrt{\beta_0}}}}{\tilde{M}}\left(\frac{2\beta_0^2}{\pi} + \frac{8\beta_0^{5/2}}{\pi^2} + \frac{2\beta_0^3(20 + \pi^2 c_1)}{\pi^3} + \frac{12\beta_0^{7/2}(20\pi^3 + \pi^5 c_1)}{\pi^7} + \frac{2\beta_0^4(840\pi^3 + 42\pi^5 c_1 + \pi^7 c_2)}{\pi^8}\right)$$

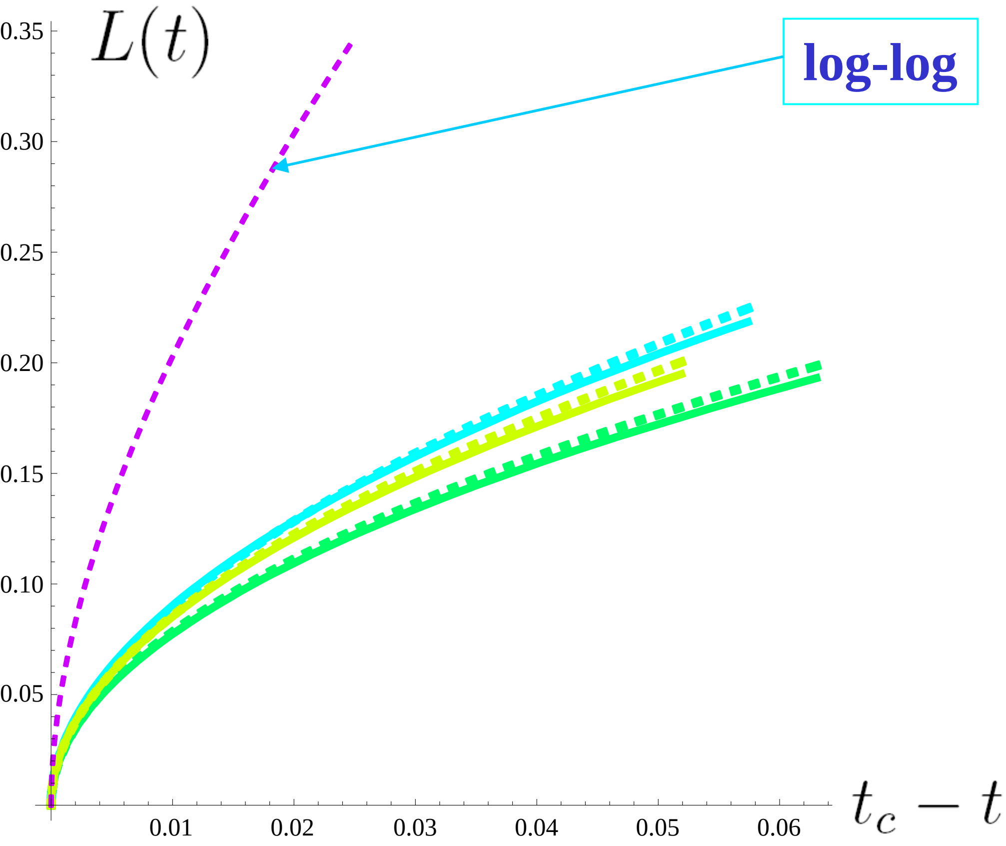
Simulations vs. analytic

$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A} \right)^{1/2}$$



Simulations vs next order analytic

$$L=\left(\frac{2\pi(t_c-t)}{\ln A-4\ln 3+4\ln\ln A}\right)^{1/2}\left[1+\frac{2(1+4\ln 3-4\ln\ln A)}{(\ln A)^2}\right.$$
$$\left.+\frac{14-48\ln\ln A+48(\ln\ln A)^2+48\ln 3-96(\ln A)(\ln 3)+48(\ln 3)^2+\frac{1}{2}\pi^2 c_1}{(\ln A)^3}+O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\right]$$

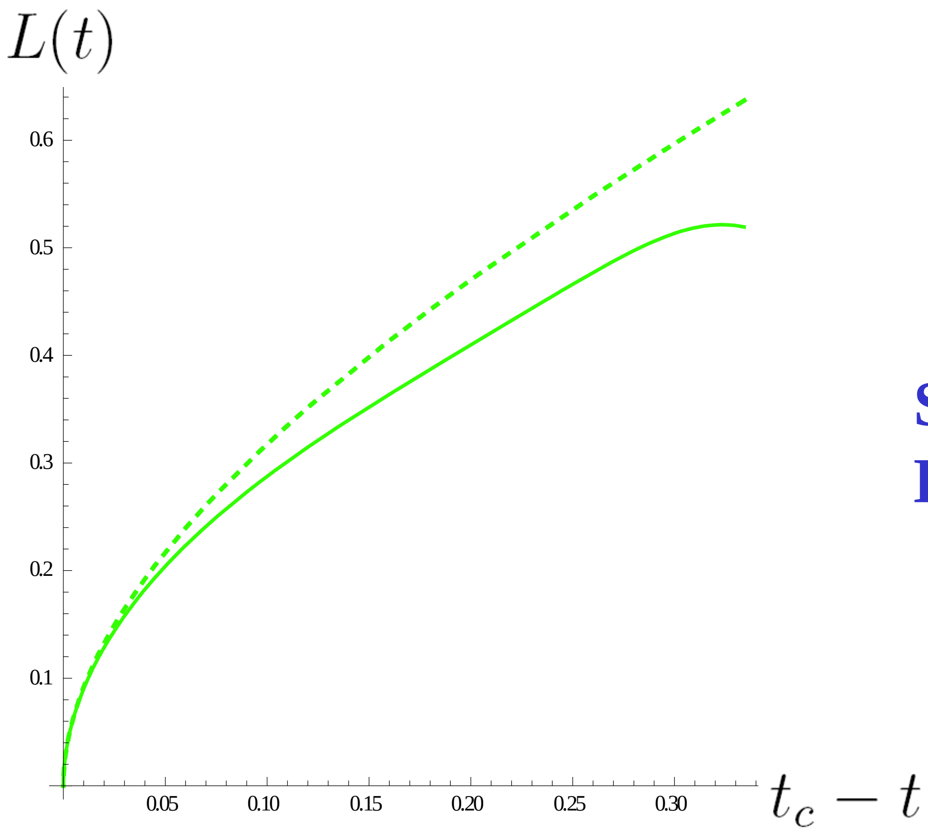


$$A=-3^4\frac{\tilde{M}}{2\pi^3}\ln\left[[2\pi(t_c-t)]^{1/2}\,\frac{e^{-a_0}}{L(z_0)}\right]$$

Solid – numerics

Dashed - analytics

Simulations vs. analytic – larger interval starting from the initial Gaussian



Solid – numerics
Dashed - analytics

In comparison, the standard log-log scaling dominates only for amplitudes above[1]

$$L=\left(2\pi\frac{t_c-t}{\ln|\ln(t_c-t)|}\right)^{1/2}$$

$$\mathbf{10^{10^{100}}} \quad = \quad \mathbf{10^{\text{Googol}}} \quad = \quad \textbf{Googolplex}$$

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[1] **P.M. Lushnikov, S.A. Dyachenko and N. Vladimirova. Physical Review A, v. 88, 013845 (2013).**